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A variational principle for magnetohydrodynamic channel flow

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A variational formulation is presented for a class of magnetohydrodynamic (MHD) channel flow problems. This formulation yields solutions for the fluid velocity and the induced electric potential in a channel with a uniform transverse static magnetic field. The channel cross-section is constant but arbitrary, and the channel walls can be either insulators or conductors with finite electrical conductivity. Electric currents are permitted to enter and leave the channel walls so that the solutions are suitable for MHD generator and pump applications. An example of a square channel with conducting walls is solved as an illustration.

1. Introduction

The study of magnetohydrodynamic (MHD) channel flow has received considerable attention in the past decade. This interest has been motivated by three principle applications: the MHD generator, the MHD pump, and the electromagnetic flowmeter.

The general model that is normally considered in these studies consists of an infinitely long channel of constant cross-section with a uniform static magnetic field applied transverse to the axis of the channel. The walls of the channel are either insulators, conductors, or a combination of insulators and conductors depending on the intended application.

For example, in the MHD generator and pump cases, the channel cross-section is normally rectangular with insulated walls perpendicular to the magnetic field and conducting walls parallel to the magnetic field. For the electromagnetic flowmeter case, the channel cross-section is normally circular with conducting walls.

In order to carry out an analytical solution for MHD channel flow it is generally necessary to make simplifying assumptions, such as requiring the channel walls to be either perfect conductors or perfect insulators or requiring the channel walls to be very thin. These and other simplifications often greatly limit the usefulness of the results. In addition, many analytical solutions give results in the form of infinite series which converge poorly for the large values of the static magnetic field that are encountered in practice.

To alleviate some of these difficulties, Tani (1962) developed a variational formulation for the solution of MHD channel flow problems. His formulation gives solutions for the velocity profile and the induced magnetic field distribution in the channel for an arbitrary channel cross-section. It requires, however, that the channel walls be either perfect conductors or insulators and that the admissible functions for the velocity and induced magnetic field satisfy appropriate boundary conditions.

In this paper a variational formulation is presented that gives solutions for the velocity profile and the electric potential distribution in a channel of arbitrary cross-section. It also gives solutions for the electric potential distribution in the channel walls. The walls of the channel can be a combination of insulators and conductors but the conductors may have a finite conductivity. Moreover, the formulation is sufficiently general to allow electric currents to enter and leave the channel walls so that the solutions obtained are suitable for the MHD generator and pump applications. The entire class of admissible functions for the velocity and potential need not satisfy the prescribed boundary conditions since they appear as the natural boundary conditions in this formulation. The class of functions must be sufficiently large, however, to contain those functions which do satisfy the boundary conditions, if the exact solution to the problem is to be obtained.

The paper concludes with an example that consists of a square channel with conducting walls of finite conductivity.

2. The model

A cross-section of a generalized channel is shown in figure 1. It consists of the fluid duct S_f bounded by the conducting walls S_c and the insulated walls S_i. The contours C_{fc} and C_{fi} denote the fluid-conducting wall interface and the fluid-insulated wall interface, respectively. The contour C_{co} denotes the outer

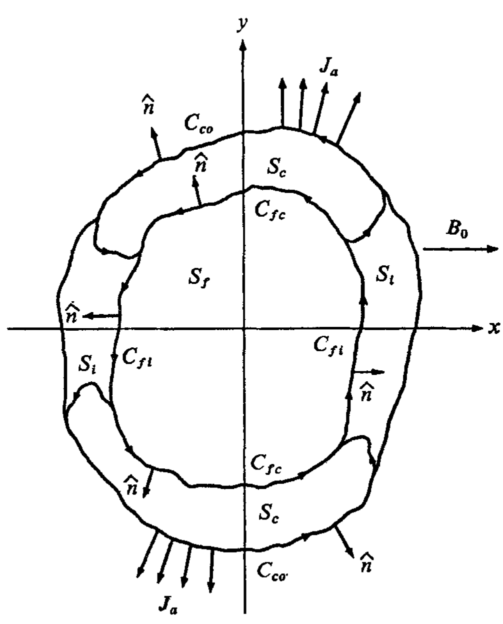

FIGURE 1. Cross-section of generalized channel.

edge of the conducting wall. The vector $\hat{n}$ is the unit normal to the contours with the positive direction as shown.

The applied static magnetic field $\mathbf{B}_0$ is uniform (independent of x, y, and z) and parallel to the x axis. The applied or generated current density at the outer edge of the conducting wall J_a is considered positive when directed outward. It is assumed that the net current entering the channel cross-section due to J_a is zero, so that the two-dimensional features of the model are retained.

3. Basic equations

The basic equations to be used are the standard MHD equations for steady state, fully developed, incompressible, laminar flow which consist of Maxwell's equations, the continuity equation, the momentum transport equation, and the generalized Ohm's law. These are:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{1a}$$

$$\nabla . \mathbf{B} = 0, \tag{1b}$$

$$\nabla . \mathbf{V} = 0, \tag{1c}$$

$$(\rho \mathbf{V} . \nabla)\, \mathbf{V} = -\nabla p + \mathbf{J} \times \mathbf{B} + \eta \nabla^2 \mathbf{V}, \tag{1d}$$

$$\mathbf{J} = \sigma_f(-\nabla U + \mathbf{V} \times \mathbf{B}), \tag{1e}$$

where U, $\mathbf{B}$, $\mathbf{J}$, and μ_0 are the electric potential, magnetic flux density, electric current density, and magnetic permeability of free-space, respectively; and $\mathbf{V}$, ρ, η, σ_f, and p are the fluid velocity, density, viscosity, electrical conductivity, and pressure, respectively. The fluid properties ρ, η, and σ_f are assumed to be constant. Equations (1a)–(1e) are based, in part, on the assumptions that the magnetic permeability of the fluid is the same as that of free space, that the convection current is negligible compared to the conduction current, and that the electrical component of the pondermotive force is negligible compared to the magnetic component.

Due to the uniformity of the channel cross-section and the applied magnetic field with respect to the z axis, it can be shown that all quantities in the basic equations are independent of z except for the pressure which is linear in z (Shercliff 1953). In addition, it can be shown that the fluid velocity $\mathbf{V}$ has only a z component V_z and that $\mathbf{J}$ has only x and y components (Hunt 1969). Furthermore, the total magnetic field $\mathbf{B}$ consists of the applied field $\mathbf{B}_0$ in the x direction and an induced field in the z direction.

Using these properties of the solution, (1a)–(1e) can be combined to give the following governing equations:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - B_0 \frac{\partial V_z}{\partial y} = 0, \tag{2a}$$

$$\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} - \frac{1}{\eta}\frac{\partial p}{\partial z} + \frac{\sigma_f B_0}{\eta}\frac{\partial U}{\partial y} - \frac{\sigma_f B_0^2}{\eta} V_z = 0. \tag{2b}$$

These equations apply, of course, only in the fluid duct region S_f. In the conducting wall region S_c, (2a) applies with $V_z = 0$. In the insulated wall region S_i,

($2a$) also applies with $V_z = 0$, but it need not be solved since the boundary conditions along the fluid-insulated wall interface permit the solution to be found exterior to the insulator region without knowing the potential within the insulator. The boundary conditions appropriate to this problem are as follows:

$$V_z|_f = 0, \quad \text{on} \quad C_{fc} \quad \text{and} \quad C_{fi}, \tag{3a}$$

$$U|_f - U|_w = 0, \quad \text{on} \quad C_{fc}, \tag{3b}$$

$$\sigma_f \nabla U . \hat{n}|_f - \sigma_w \nabla U . \hat{n}|_w = 0, \quad \text{on} \quad C_{fc}, \tag{3c}$$

$$\sigma_f \nabla U . \hat{n}|_f = 0, \quad \text{on} \quad C_{fi}, \tag{3d}$$

$$\sigma_w \nabla U . \hat{n}|_w + J_a = 0, \quad \text{on} \quad C_{co}, \tag{3e}$$

where $|_f$ and $|_w$ refer to evaluating the quantity on the fluid or wall side of the contour, respectively, and where σ_w is the conductivity of the conducting walls.

The boundary condition equations ($3a$)–($3e$) require the following:

(i) The fluid velocity must vanish on the fluid–wall interfaces C_{fc} and C_{fi}.

(ii) The electric potential must be continuous across the fluid-conducting wall interface C_{fc}.

(iii) The component of the electric current normal to the fluid-conducting wall interface C_{fc} must be continuous.

(iv) The component of the electric current normal to the fluid-insulated wall interface C_{fi} must vanish.

(v) The component of the electric current normal to the outer edge of the conducting wall C_{co} must equal the applied or generated current J_a.

In solving the equations it is convenient to work with dimensionless quantities. This can easily be accomplished by defining L and V_0 to be a characteristic dimension and characteristic velocity of the channel. Let

$$X = x/L, \quad Y = y/L, \quad Z = z/L \quad \text{(dimensionless co-ordinates)}, \tag{4a}$$

$$W = U/B_0 L V_0 \text{ (dimensionless potential)}, \tag{4b}$$

$$V = V_z/V_0 \text{ (dimensionless velocity)}, \tag{4c}$$

$$M = B_0 L(\sigma_f/\eta)^{\frac{1}{2}} \text{ (Hartmann number)}, \tag{4d}$$

$$P_0 = \frac{-L^2}{\eta V_0} \frac{\partial p}{\partial z} \text{ (dimensionless pressure gradient)}, \tag{4e}$$

$$J_0 = J_a/B_0 V_0 \sigma_w \text{ (dimensionless applied or generated current)}, \tag{4f}$$

$$\gamma = \sigma_w/\sigma_f \text{ (ratio of wall-to-fluid conductivity)}. \tag{4g}$$

Combining ($4a$)–($4g$) with ($2a$)–($2b$) yields the following set of equations in dimensionless form:

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial V}{\partial Y} = 0, \quad \text{on} \quad S_f, \tag{5a}$$

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + P_0 + M^2 \frac{\partial W}{\partial Y} - M^2 V = 0, \quad \text{on} \quad S_f, \tag{5b}$$

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = 0, \quad \text{on} \quad S_c. \tag{5c}$$

Likewise, combining (4*a*)–(4*g*) with (3*a*)–(3*e*) gives the following set of dimensionless boundary condition equations:

$$V|_f = 0, \quad \text{on} \quad C_{fc} \quad \text{and} \quad C_{fi}, \tag{5d}$$

$$W|_f - W|_w = 0, \quad \text{on} \quad C_{fc}, \tag{5e}$$

$$\frac{1}{[1+(dY/dX)^2]^{\frac{1}{2}}}\left\{\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right\}\bigg|_f - \frac{\gamma}{[1+(dY/dX)^2]^{\frac{1}{2}}} \times \left\{\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right\}\bigg|_w = 0, \quad \text{on} \quad C_{fc}, \tag{5f}$$

$$\frac{1}{[1+(dY/dX)^2]^{\frac{1}{2}}}\left\{\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right\}\bigg|_f = 0, \quad \text{on} \quad C_{fi}, \tag{5g}$$

$$\frac{1}{[1+(dY/dX)^2]^{\frac{1}{2}}}\left\{\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right\}\bigg|_w - J_0 = 0, \quad \text{on} \quad C_{co}, \tag{5h}$$

The unit normal vector $\hat{n}$ has been replaced by

$$\hat{n} = \frac{-\dfrac{dY}{dX}\hat{a}_x + \hat{a}_y}{[1+(dY/dX)^2]^{\frac{1}{2}}}, \tag{6}$$

where $\hat{a}_x$ and $\hat{a}_y$ are the unit vectors in the X and Y directions, respectively. The sign of the square root must be selected so that the positive direction for $\hat{n}$ is as shown in figure 1.

4. Variational expression

To convert the solution of the governing differential equations into an equivalent variational problem, a functional of the dependent variables V and W must be constructed so that the associated Euler–Lagrange equations are the basic governing equations (5*a*)–(5*c*), and the corresponding natural boundary conditions are the prescribed boundary condition equations (5*d*)–(5*h*). This construction will be performed by summing terms that are obtained by multiplying each governing equation and boundary condition equation by a suitable function and then integrating over the corresponding area or contour where the equation is valid.

Let δV and δW be the variations of V and W, respectively, where it is assumed that δV and δW are continuous with piecewise continuous first derivatives. The integrals

$$I_1 \equiv 2\int_{S_f}\left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + P_0 + M^2\frac{\partial W}{\partial Y} - M^2 V\right]\delta V\, dX\, dY, \tag{7a}$$

$$I_2 \equiv 2M^2\int_{S_f}\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial V}{\partial Y}\right]\delta W\, dX\, dY, \tag{7b}$$

$$I_3 \equiv 2\gamma M^2\int_{S_c}\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right]\delta W\, dX\, dY, \tag{7c}$$

are identically zero for any δV and δW, since the quantities in square brackets are identically zero by virtue of (5*a*)–(5*c*).

Four additional integral expressions that are identically zero can be obtained from (5*d*)–(5*h*) by integrating along appropriate contours. Recalling that the differential length along a contour is given by $[1+(dY/dX)^2]^{\frac{1}{2}}dX$, these integral expressions can be defined as

$$I_4 \equiv -2M^2\int_{C_{fc}} \left[\left(\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX\right)\Big|_f - \gamma\left(\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX\right)\Big|_w\right]\delta W, \quad (7d)$$

$$I_5 \equiv -2M^2\int_{C_{fi}} \left[\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX\right]\Big|_f \delta W, \quad (7e)$$

$$I_6 \equiv -2\gamma M^2\int_{C_{eo}} \left[\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX - J_0[1+(dY/dX)^2]^{\frac{1}{2}}dX\right]\Big|_w \delta W, \quad (7f)$$

$$I_7 \equiv 2\int_{C_{fc}+C_{fi}} [V]\left(\frac{\partial \delta V}{\partial X}dY - \frac{\partial \delta V}{\partial Y}dX - M^2\delta W\,dX\right)\Big|_f. \quad (7g)$$

These integrals vanish since the quantities in brackets are zero by virtue of (5*d*)–(5*h*).

The integrals I_1–I_7 can be integrated by parts and combined to give

$$\sum_{n=1}^{7} I_n = \delta F, \quad (8a)$$

where

$$\begin{aligned} F \equiv & \int_{S_f}\left\{2P_0V - \left(\frac{\partial V}{\partial X}\right)^2 - \left(\frac{\partial V}{\partial Y}\right)^2 - M^2\left[V^2 + \left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2 - 2V\frac{\partial W}{\partial Y}\right]\right\}dX\,dY \\ & -\gamma M^2\int_{S_c}\left\{\left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2\right\}dX\,dY + 2\int_{C_{fc}+C_{fi}} V\left\{\frac{\partial V}{\partial X}dY - \frac{\partial V}{\partial Y}dX\right\}\Big|_f \\ & + 2\gamma M^2\int_{C_{eo}} WJ_0[1+(dY/dX)^2]^{\frac{1}{2}}\,|_w\,dX. \end{aligned} \quad (8b)$$

Since each I_n $(n = 1, \ldots, 7)$ is identically zero, the variation of the functional F is zero. Thus, F is stationary; that is, first-order changes in V and W about their true values produce only second-order changes in F. The converse of this statement is also true. Of all functions that are continuous with piecewise continuous first derivatives, the particular pair of functions for V and W that make F stationary satisfy both (5*a*)–(5*c*) and (5*d*)–(5*h*) and, hence, are the desired solutions.

Even though F was shown to be stationary, it does not necessarily mean that F has an extremum at the true solution for V and W. For example, as V and W are varied from their true values, F may always increase, always decrease, or either increase or decrease depending on how V and W are varied. To determine which case corresponds to the F under consideration, the quantity $F(V+\delta V, W+\delta W) - F(V, W)$ is computed, giving

$$\begin{aligned} & F(V+\delta V, W+\delta W) - F(V,W) \\ & = 2\int_{S_f}\left\{\left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + P_0 - M^2V + M^2\frac{\partial W}{\partial Y}\right]\delta V \right. \\ & \left. + M^2\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial V}{\partial Y}\right]\delta W\right\}dX\,dY + 2\gamma M^2\int_{S_c}\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right]\delta W\,dX\,dY \end{aligned}$$

$$-2M^2\int_{C_{fc}}\left[\left(\frac{\partial W}{\partial X}dY-\frac{\partial W}{\partial Y}dX\right)\Big|_f-\gamma\left(\frac{\partial W}{\partial X}dY-\frac{\partial W}{\partial Y}dX\right)\Big|_w\right]\delta W$$

$$-2M^2\int_{C_{fi}}\left[\frac{\partial W}{\partial X}dY-\frac{\partial W}{\partial Y}dX\right]\delta W\Big|_f-2\gamma M^2\int_{C_{co}}\left[\left(\frac{\partial W}{\partial X}dY-\frac{\partial W}{\partial Y}dX\right)\right.$$

$$\left.-J_0[1+(dY/dX)^2]^{\frac{1}{2}}dX\right]\delta W\Big|_w$$

$$+2\int_{C_{fc}+C_{fi}}[V]\left(\frac{\partial\delta V}{\partial X}dY-\frac{\partial \delta V}{\partial Y}dX-M^2\delta W\,dX\right)\Big|_f$$

$$-\int_{S_f}\left[\left(\frac{\partial\delta V}{\partial X}\right)^2+\left(\frac{\partial\delta V}{\partial Y}\right)^2+M^2\left(\frac{\partial\delta W}{\partial X}\right)^2+M^2\left(\frac{\partial\delta W}{\partial Y}+\delta V\right)^2\right]dX\,dY$$

$$-\gamma M^2\int_{S_c}\left[\left(\frac{\partial\delta W}{\partial X}\right)^2+\left(\frac{\partial\delta W}{\partial Y}\right)^2\right]dX\,dY+2\int_{C_{fc}+C_{fi}}\left[\frac{\partial\delta V}{\partial X}dY-\frac{\partial\delta V}{\partial Y}dX\right]\delta V\Big|_f. \tag{9}$$

A careful examination of (9) reveals that the first six integrals vanish because the quantities in large square brackets are identically zero. The remaining integrals are of second-order in δV and δW. This result is not surprising, since F was constructed so that all first-order terms in δV and δW vanished. It is easily seen that (5*a*)–(5*c*) are the associated Euler–Lagrange equations of the functional F and that (5*d*)–(5*h*) are the natural boundary conditions. Of the three remaining integrals in (9), two are negative definite and the last can be of either sign. Thus, F has neither a maximum nor minimum at the true values for V and W. However, if the class of admissible functions for V and W is restricted so that the last integral must vanish, then F corresponds to a maximum at the true values for V and W, since $F(V+\delta V, W+\delta W)-F(V,W)\leqslant 0$.

A study of the last integral in (9) reveals that the proper restriction to impose on the class of functions for V is that each function must vanish on the contours C_{fc} and C_{fi}. An alternate choice which also makes the last integral vanish is to specify the normal derivative for each member of the class on C_{fc} and C_{fi}. This choice is useless, however, since it would require solving the problem another way first to determine the correct value for the normal derivative.

Requiring the entire class of functions for V to vanish on C_{fc} and C_{fi} may provide a great simplification in many problems in obtaining approximate values for V and W since finding a maximum for F is often much easier than finding a stationary point. Moreover, requiring the entire class of functions for V to vanish on C_{fc} and C_{fi} completely eliminates one integral in the expression for F given by (8*b*).

Before a solution for V and W can be determined, values for P_0 and J_0 must be specified. The dimensionless pressure gradient P_0 must be a constant. The dimensionless current density J_0, however, can be specified as a function of the coordinates along the contour C_{co}. Since the basic equations and boundary condition equations are linear in V, W, P_0, and J_0 solutions for V and W can be obtained by superimposing solutions for $P_0\neq 0$ and $J_0=0$ with those for $P_0=0$ and $J_0\neq 0$.

To complete the study of the variational expression, it is desirable to determine its physical significance.

5. Physical significance of variational expression

Consider the power or energy balance that exists in MHD channel flow. The power per unit length that is supplied to the fluid by the pressure gradient $P_{\Delta p}$ is given by

$$P_{\Delta p} = -\int_{S_f} \frac{\partial p}{\partial z} V_z \, dx \, dy = \eta V_0^2 \int_{S_f} P_0 V \, dX \, dY. \tag{10a}$$

If this quantity is negative, it simply means that the channel is acting as a pump.

The dissipative terms consist of the viscous losses in the fluid P_η, the ohmic losses in the fluid P_{σ_f}, and the ohmic losses in the conducting walls P_{σ_w}. Expressing each of these in terms of the power dissipated per unit length along the channel gives

$$P_\eta = -\int_{S_f} \eta \left[\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2}\right] V_z \, dx \, dy = \eta V_0^2 \int_{S_f} \left[\left(\frac{\partial V}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2\right] dX \, dY$$

$$-\eta V_0^2 \int_{C_{fc}+C_{fi}} V \left[\frac{\partial V}{\partial X} dY - \frac{\partial V}{\partial Y} dX\right]\Bigg|_f, \tag{10b}$$

$$P_{\sigma_f} = \sigma_f \int_{S_f} \left| -\frac{\partial U}{\partial x}\hat{a}_x - \frac{\partial U}{\partial y}\hat{a}_y + V_z B_0 \hat{a}_y \right|^2 dx \, dy$$

$$= \eta V_0^2 M^2 \int_{S_f} \left[\left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y} - V\right)^2\right] dX \, dY, \tag{10c}$$

$$P_{\sigma_w} = \sigma_w \int_{S_c} \left| -\frac{\partial U}{\partial x}\hat{a}_x - \frac{\partial U}{\partial y}\hat{a}_y \right|^2 dx \, dy$$

$$= \eta V_0^2 \gamma M^2 \int_{S_c} \left[\left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2\right] dX \, dY. \tag{10d}$$

Equation (10*b*) reveals that the viscous losses in the fluid can be split into two parts: the volume losses $P_{\eta v}$ and the surface losses $P_{\eta s}$, where

$$P_{\eta v} = \eta V_0^2 \int_{S_f} \left[\left(\frac{\partial V}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2\right] dX \, dY, \tag{10e}$$

$$P_{\eta s} = -\eta V_0^2 \int_{C_{fc}+C_{fi}} V \left[\frac{\partial V}{\partial X} dY - \frac{\partial V}{\partial Y} dX\right]\Bigg|_f. \tag{10f}$$

The surface losses are zero when the boundary condition is imposed that requires V to vanish on C_{fc} and C_{fi}.

The remaining term to be considered is the loss due to the current J_a. Since J_a is positive by definition when it is directed outward, the power that is supplied to an external load per unit length of the channel P_{J_a} is given by

$$P_{J_a} = -\int_{C_{co}} U J_a [1 + (dy/dx)^2]^{\frac{1}{2}} dx = -\eta V_0^2 \gamma M^2 \int_{C_{co}} W J_0 [1 + (dY/dX)^2]^{\frac{1}{2}} dX. \tag{10g}$$

If P_{J_a} is negative, it simply indicates that power is being supplied by an external source.

Since power is conserved, the power balance for the channel can be expressed as

$$P_{\Delta p} = P_{\eta v} + P_{\eta s} + P_{\sigma f} + P_{\sigma w} + P_{J_a}. \tag{11}$$

Comparing the expression for the functional F given by (8*b*) with the various power dissipation terms given by (10*a*)–(10*g*) reveals that F can be expressed as

$$\eta V_0^2 F = 2P_{\Delta p} - P_{\eta v} - P_{\sigma f} - P_{\sigma w} - 2P_{\eta s} - 2P_{J_a}. \tag{12}$$

A word of caution is in order at this point. The expression for F given by (8*b*) is defined and valid for an arbitrary choice for V and W. Likewise, the power dissipation terms $P_{\eta v}$, $P_{\sigma f}$, etc., given by (10*a*)–(10*g*), are valid for arbitrary values for V and W. Thus, (12) is valid, in general. However, (11), which is the power balance for the channel, is only valid, in general, for the correct solutions for V and W.

As shown in § 4, the stationary point for F corresponds to the true solutions for V and W. For these values only, (11) and (12) can be combined to give

$$F_{st} = (P_{\Delta p} - P_{J_a})/\eta V_0^2, \tag{13}$$

where it has been recognized that $P_{\eta s}$ vanishes for the true V. Thus, the stationary value of F is proportional to the difference between the power supplied to the fluid by the pressure gradient and the electrical power delivered to an external load.

An important special case occurs for $J_a = 0$, which yields

$$F_{st} = \int_{S_f} P_0 V \, dX \, dY, \tag{14}$$

where $P_{\Delta p}$ has been replaced using (10*a*). Since the dimensionless pressure gradient P_0 is a constant, the stationary value for F is proportional to the average fluid velocity in the channel. This is a very important result since the average velocity, which is often the main quantity of interest, is proportional to a stationary quantity which can be computed to good accuracy.

If the boundary condition $V = 0$ on C_{fc} and C_{fi} is satisfied by all admissible functions, then F has a maximum at its stationary point as noted in the previous section. The maximum for F using a subset of the class of admissible functions for V and W will be less than or equal to the maximum for F using the entire class of admissible functions. Thus, a lower limit for the average velocity can be easily found by using any admissible function.

6. Example: square channel with conducting walls

A square channel is shown in figure 2 using the dimensionless co-ordinates. The characteristic dimension for the channel L has been chosen as one-half its height or width. The normalized wall thickness t is the actual wall thickness divided by L.

Approximate solutions for the velocity and electric potential will be obtained using the Ritz technique. In this technique, the velocity and potential are expressed in terms of known functions of X and Y that approximate the true solution but contain adjustable parameters $\lambda_1, \ldots, \lambda_n$. The approximate solutions for V and W are then substituted into the expression for F given by (8*b*), and the indicated integrations with respect to X and Y are performed. This leaves F as a function of the parameters $\lambda_1, \ldots, \lambda_n$ and the characteristic parameters of the channel P_0, γ, M, and J_0. Assuming that the approximate solution for V vanishes at $X = \pm 1$ and $Y = \pm 1$ for all values of $\lambda_1, \ldots, \lambda_n$, the stationary value of F can be found by maximizing F with respect to $\lambda_1, \ldots, \lambda_n$. The corresponding values for $\lambda_1, \ldots, \lambda_n$ when substituted into the approximate functions for V and W will yield the closest approximations to the velocity and potential that are possible for the class of functions used.

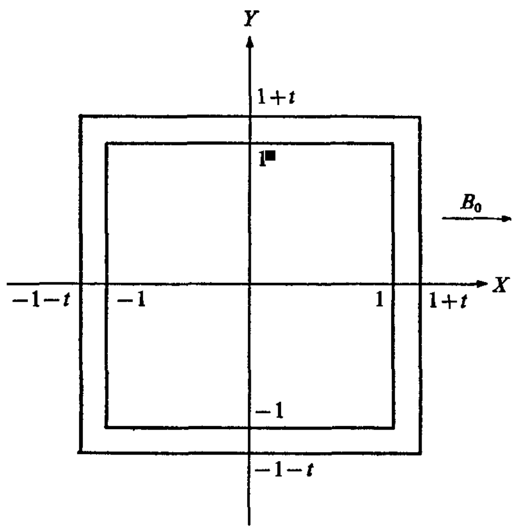

FIGURE 2. Cross-section of square channel with conducting walls.

The solution for the square channel will be determined for $P_0 = 1$ and $J_0 = 0$. Let V and W be approximated by the trial functions given by

$$V(X,Y) \doteq A_1(1-X^{\alpha_1})(1-Y^{\alpha_2}) \quad (0 \leqslant X \leqslant 1,\ 0 \leqslant Y \leqslant 1), \tag{15a}$$

$$W(X,Y) \doteq (C_1 Y^{\beta_1} + C_2 Y^{\beta_2})(1+C_3 X^{\beta_3}) \quad (0 \leqslant X \leqslant 1+t,\ 0 \leqslant Y \leqslant 1+t), \tag{15b}$$

where A_1, C_1, C_2, C_3, α_1, α_2, β_1, β_2, and β_3 are adjustable parameters. Because of the symmetry of the problem it is only necessary to specify V and W in the first quadrant. For other quadrants, V and W can be found using the relations

$$V(X,Y) = V(X,-Y) = V(-X,Y) \text{ and } W(X,Y) = W(-X,Y) = -W(X,-Y).$$

Since the admissible functions for V and W must be continuous with piecewise continuous derivatives, all exponents in (15*a*)–(15*b*) must be one or greater.

Trial functions of the form given by (15*a*) have proved to be very useful in obtaining approximate solutions for problems of this class. The functions offer great flexibility with a minimum number of parameters since the velocity profile, for example, can go from a parabolic profile ($\alpha_1 = \alpha_2 = 2$) to nearly slug flow (α_1, α_2 large) by merely varying the two exponent parameters α_1 and α_2.

Substituting (15*a*)–(15*b*) into the expressions for F given by (8*b*) yields, after performing the integrations,

$$F = [\psi]^T[A][\psi] - [\psi]^T[D], \tag{16}$$

where
$$[\psi] = \begin{bmatrix} A_1 \\ C_1 \\ C_2 \end{bmatrix}, \quad [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad [D] = \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix},$$

$$a_{11} = -\frac{8\alpha_1^2\alpha_2^2}{(\alpha_2+1)(2\alpha_1-1)(2\alpha_2+1)} - \frac{8\alpha_1^2\alpha_2^2}{(\alpha_1+1)(2\alpha_2-1)(2\alpha_1+1)} - \frac{16M^2\alpha_1^2\alpha_2^2}{(\alpha_1+1)(2\alpha_1+1)(\alpha_2+1)(2\alpha_2+1)},$$

$$a_{12} = \frac{4M^2\alpha_1\alpha_2}{\beta_1+\alpha_2}\left[\frac{1}{\alpha_1+1} + \frac{C_3}{(\beta_3+1)(\beta_3+\alpha_1+1)}\right],$$

$$a_{13} = \frac{4M^2\alpha_1\alpha_2}{\beta_2+\alpha_2}\left[\frac{1}{\alpha_1+1} + \frac{C_3}{(\beta_3+1)(\beta_3+\alpha_1+1)}\right],$$

$$a_{22} = -4M^2\left[\frac{\beta_3^2C_3^2}{(2\beta_3-1)(2\beta_1+1)} + \frac{\beta_1^2}{2\beta_1-1}\left(1+\frac{2C_3}{\beta_3+1}+\frac{C_3^2}{2\beta_3+1}\right) + \gamma t\left\{\frac{2(\beta_1+\beta_3)\beta_3^2C_3^2}{(2\beta_3-1)(2\beta_1+1)} + \frac{2\beta_1^2}{2\beta_1-1}\left(\beta_1+\frac{(2\beta_1+\beta_3)C_3}{\beta_3+1}+\frac{(\beta_1+\beta_3)C_3^2}{2\beta_3+1}\right)\right\}\right],$$

$$a_{23} = -4M^2\left[\frac{\beta_3^2C_3^2}{(2\beta_3-1)(\beta_1+\beta_2+1)} + \frac{\beta_1\beta_2}{\beta_1+\beta_2-1}\left(1+\frac{2C_3}{\beta_3+1}+\frac{C_3^2}{2\beta_3+1}\right) + \gamma t\left\{\frac{(\beta_1+\beta_2+2\beta_3)\beta_3^2C_3^2}{(2\beta_3-1)(\beta_1+\beta_2+1)} + \frac{\beta_1\beta_2}{\beta_1+\beta_2-1}\left((\beta_1+\beta_2)+\frac{2(\beta_1+\beta_2+\beta_3)C_3}{\beta_3+1} + \frac{(\beta_1+\beta_2+2\beta_3)C_3^2}{2\beta_3+1}\right)\right\}\right],$$

$$a_{33} = -4M^2\left[\frac{\beta_3^2C_3^2}{(2\beta_3-1)(2\beta_2+1)} + \frac{\beta_2^2}{2\beta_2-1}\left(1+\frac{2C_3}{\beta_3+1}+\frac{C_3^2}{2\beta_3+1}\right) + \gamma t\left\{\frac{2(\beta_2+\beta_3)\beta_3^2C_3^2}{(2\beta_3-1)(2\beta_2+1)} + \frac{2\beta_2^2}{2\beta_2-1}\left(\beta_2+\frac{(2\beta_2+\beta_3)C_3}{\beta_3+1}+\frac{(\beta_2+\beta_3)C_3^2}{2\beta_3+1}\right)\right\}\right],$$

$$d_1 = -\frac{8\alpha_1\alpha_2}{(\alpha_1+1)(\alpha_2+1)}.$$

In obtaining (16) the 'thin wall' approximation $t \ll 1$ was made so that the results are directly comparable with published values. This approximation did not have to be made in order to use this technique but was done since only thin-wall results are available for comparison.

The maximum value for F and the corresponding values for the parameters were found using a computer. Since the normalized cross-sectional area of the channel is 4 and $P_0 = 1$, the proportionality constant between the average normalized fluid velocity $\bar{V}$ and F_{st} is $\frac{1}{4}$ (see (14)). Values for $\bar{V}$ as a function of the Hartmann number M are shown in figure 3 for various values of conduction parameter γt.

The variational solution can be compared with other solutions for some limiting cases. For $M = 0$, the average dimensionless velocity from the variational solution is 0·1403, independent of γt as compared with the exact value of 0·1406, which can be computed using Fourier expansion techniques.

The exact solution for the average flow in a rectangular channel with insulated walls ($\gamma t = 0$) and perfectly conducting walls ($\gamma t = \infty$) has been obtained by Shercliff (1953) and Chang & Lundgren (1961), respectively. Each of these

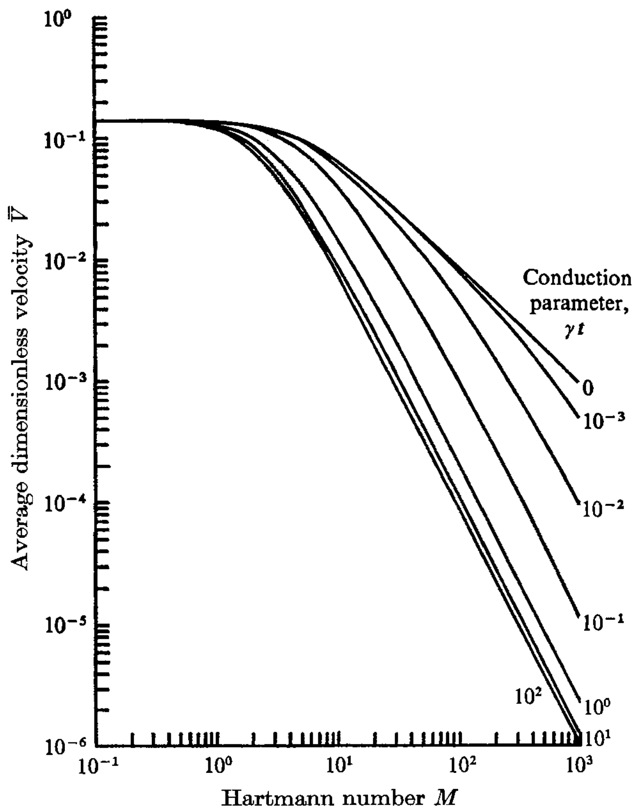

FIGURE 3. Variational solution for average dimensionless velocity in square channel.

solutions is in the form of a series which converges poorly for large M. Williams (1963), however, transformed these solutions and obtained asymptotic forms for the average velocity for large M. These solutions, simplified for $P_0 = 1$ and the square channel, are as follows:

$$\bar{V} = \frac{1}{M}\left[1 - \frac{32}{15(2\pi M)^{\frac{1}{2}}} - \frac{1}{M} + \frac{4}{3(2\pi M^3)^{\frac{1}{2}}} + \theta(1/M^2)\right], \quad \gamma t = 0,$$

$$\bar{V} = \frac{1}{M^2}\left[1 - \frac{1}{M} - \frac{1}{M^{\frac{3}{2}}}\left\{2\cdot 43 - \frac{3\cdot 03}{M^{\frac{1}{2}}} + \frac{0\cdot 48}{M}\right\} + \theta(1/M^3)\right], \quad \gamma t = \infty.$$

A comparison between the variational solutions and these asymptotic forms is shown in figure 4. The agreement between the two solutions for $\gamma t = 0$ is excellent for $M \geqslant 10$. The variational solution for $\gamma t = 10^4$ is always slightly less than the asymptotic form for $\gamma t = \infty$. The difference, however, decreases to less than 0·1 % at $M = 1000$.

A Fourier expansion type solution for the rectangular channel with 'thin walls' of finite conductivity has recently been obtained by Chu (1969). A comparison of his solution with the variational solution is shown in figure 5. The

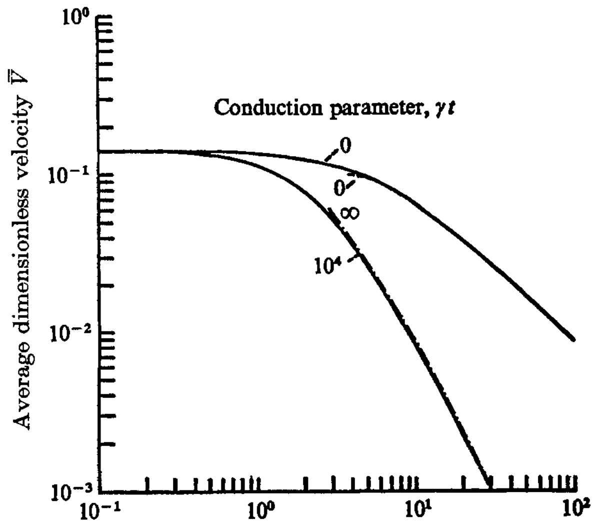

FIGURE 4. Comparison of variational solution with Williams (1963) asymptotic solutions for average dimensionless velocity: ——, variational; ---, Williams for $\gamma t = 0$; —·—, Williams for $\gamma t = \infty$.

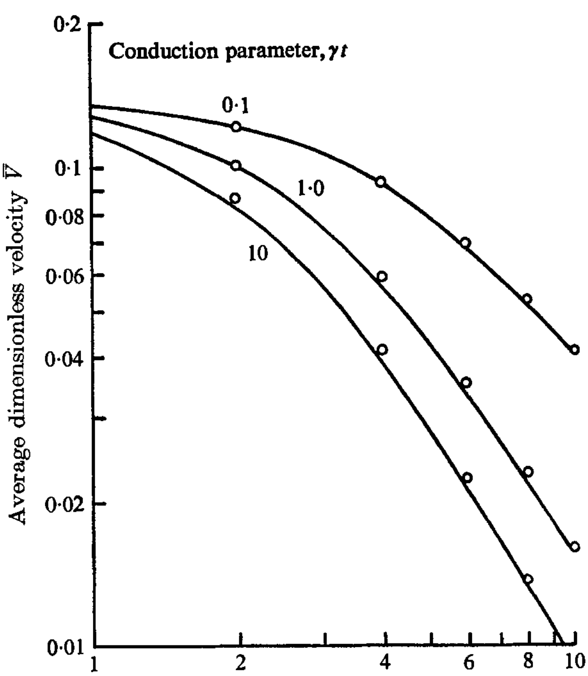

FIGURE 5. Comparison of variational solution with Chu's (1969) solution for average dimensionless velocity: ——, variational; ○, Chu.

agreement between these two solutions is also quite good. As shown, the variational solution for the average velocity is always slightly less than the series solution value. This is due to the fact that the computed maximum for F, and hence the average velocity, is always less than or equal to the true maximum for F since the trial functions used are a subset of the entire class of admissible functions.

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